Two-Stream Instability in Finite Beams

E. A. FRIEMAN AND M. L. GOLDBERGER
Princeton University, Princeton, New Jersey
K. M. WATSON AND S. WEINBERG
University of California, Berkeley, California
AND
M. N. ROSENBLUTH
University of California, La Jolla, California and General Atomic Division, General Dynamics Corporation
(Received September 5, 1961)

The streaming instabilities of a finite beam of charged particles passing through a zero-temperature plasma are studied. It is shown that there are no eigenmodes associated with the instabilities. Nevertheless, by constructing wave-packet disturbances one is led to instabilities similar to those for a beam of infinite extent.

I. INTRODUCTION

THERE are a number of instabilities that can afflict a beam of relativistic particles injected into a plasma. We will be concerned here with the "electrostatic" or "two-stream" mode,[1-3] and shall discuss in particular the effects associated with the finite size and nonuniformity of the beam.[4]

The background plasma is supposed to be infinite and uniform, the beam traveling through it with uniform velocity v in the z direction. For the most part it will be assumed that the density of charged particles in the beam is very much less than the density of electron-ion pairs in the plasma, and that the disturbance in question is small, and almost a pure longitudinal plasma oscillation. We will neglect effects due to the plasma and beam temperatures or to collisions involving beam particles, but we will take into account collisions between plasma particles.[5]

[The conditions under which the thermal energy of the beam may be neglected are

$$\mathrm{Im}\,(\omega) \gg k_{\parallel}(\Delta v_{\parallel}),$$
$$\mathrm{Im}\,(\omega) \gg k_{\perp}(\Delta v_{\perp}).$$

Here Im (ω) is the growth rate of the instability, $k_{\parallel}$ and $k_{\perp}$ are the respective wave numbers parallel to and perpendicular to the beam axis, and $\Delta v_{\parallel}$ and $\Delta v_{\perp}$ are the corresponding thermal speeds of the beam particles. The condition for neglecting the finite plasma temperature is

$$\mathrm{Im}\,(\omega) \gg (k_{\parallel}a)^2/\omega_P,$$

where a is the thermal electron speed in the plasma and ω_P is the plasma frequency.]

It will be seen that the Fourier components of the electric field associated with this instability must be singular somewhere within the beam edge. For this reason the problem here is far less straightforward than might be supposed.

The fundamental differential equations for the electric field are derived in Sec. II, and simplified by using the "weak-beam" approximation in Sec. III. The necessity of a singularity within the beam edge is proven in Sec. IV. In Sec. V we discuss the time dependence expected. Section VI is devoted to a proof that in general the instability frequencies form a continuum, with no "quantization"; however, we show in Sec. VII that the spectrum is effectively discrete for a sharp enough beam edge. Some general features of linear instability theory which are needed in Secs. IV and V are discussed in detail in an appendix.

Our main result is that the worst instabilities of a finite beam are essentially the same as those of an infinite beam. Some of the less rapidly growing modes may be absent or widely spaced for a finite beam.

II. DIFFERENTIAL EQUATION

The instability is described by an electric field $\mathbf{E}$, with time and $\mathbf{r}$ dependence given by factors $e^{-i\omega t}$ and e^{ikz}. (For the present we will not assume circular symmetry about the z axis.) The fundamental equation for $\mathbf{E}$ is

$$-c^2\nabla \times (\nabla \times \mathbf{E}) = -4\pi i\omega(\mathbf{J}^B + \mathbf{J}^P). \qquad (1)$$

[1] M. A. Lampert, J. Appl. Phys. **27**, 5 (1956); O. Buneman, Phys. Rev. Letters **1**, 104 (1958); G. I. Budker, Atomnaya Energ. **5**, 9 (1956).

[2] J. Drummond, *Plasma Physics* (McGraw-Hill Book Company, Inc., New York, 1961).

[3] S. A. Bludman, K. M. Watson, and M. N. Rosenbluth, Phys. Fluids **3**, 747 (1960); H. Lewis, Institute for Defense Analyses (unpublished report).

[4] The present discussion represents a direct generalization of reference 3 to the case of a nonuniform beam. Reference 3 should be consulted for a discussion of the physical basis for the equations used here.

[5] The conditions for the validity of this neglect of beam and plasma temperature are discussed in reference 3.

In order to obtain the plasma current $\mathbf{J}^P$ and beam current $\mathbf{J}^B$ in terms of $\mathbf{E}$ we assume that

$$-4\pi i\omega \mathbf{J}^P = \omega_P^2 \mathbf{E}, \tag{2}$$

$$-4\pi i\omega' \mathbf{J}^{B'} = (\omega_B^2)'\mathbf{E}'. \tag{3}$$

The first equation refers to quantities measured in the "lab" frame, in which the plasma is at rest, while the primes in the second equation imply that the quantities refer to the "beam" frame, which travels with the beam at velocity v.

The coefficient ω_P^2 is given approximately in terms of the plasma electron density n_P, electron mass m_e, and electron-ion collision frequency ω_c, by

$$\omega_P^2 = \frac{4\pi n_P e^2}{m_e}\frac{\omega}{\omega + i\omega_c} \equiv \omega_R^2 \frac{\omega}{\omega + i\omega_c}, \tag{4}$$

and ω_B^2 can similarly be expressed in terms of the beam density n_B and beam particle mass m_B as

$$\omega_B^2 = 4\pi n_B e^2/m_B. \tag{5}$$

Since n_P and n_B may vary from point to point in the (x, y) plane, ω_P^2 and ω_B^2 are not generally constant. Later we will make the assumptions that $\omega_B'^2 \ll \omega_P^2$, and that the nonconstant part of ω_P^2 is at most of order $\omega_B'^2$.

In order to use (3) we must rewrite it in terms of quantities in the "lab" frame, using the Lorentz transformation equations

$$J_z^{B'} = (\gamma/\omega)(\Omega J_z^B + iv\nabla_\perp\cdot\mathbf{J}_\perp^B), \tag{6}$$

$$\mathbf{J}_\perp^{B'} = \mathbf{J}_\perp^B, \tag{7}$$

$$E_z' = E_z, \tag{8}$$

$$\mathbf{E}_\perp' = (\gamma/\omega)(\Omega\mathbf{E}_\perp - iv\nabla_\perp E_z), \tag{9}$$

$$\omega' = \gamma\Omega, \tag{10}$$

$$\omega_B'^2 = \omega_B^2/\gamma \equiv \omega_T^2, \tag{11}$$

where as usual $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta = v/c$, and

$$\Omega = \omega - kv. \tag{12}$$

The subscript $\perp$ refers to the part of a vector perpendicular to the z axis. [We have used the conservation of current to eliminate the charge density from (6), and Faraday's law to eliminate the magnetic field from (9).] In the "lab" frame we now have

$$-4\pi i\omega\mathbf{J}_\perp^B = \omega_T^2[\mathbf{E}_\perp - (iv/\Omega)\nabla_\perp E_z], \tag{13}$$

$$-4\pi i\omega J_z^B = \frac{\omega_T^2\omega^2}{\gamma^2\Omega^2}E_z - \frac{iv}{\Omega}\nabla_\perp\omega_T^2\left(\mathbf{E}_\perp - \frac{iv}{\Omega}\nabla_\perp E_z\right). \tag{14}$$

Assembling (1), (2), (13), and (14), we obtain the exact differential equations of our problem

$$-\nabla_\perp(\nabla_\perp\cdot\mathbf{E}_\perp) + \nabla_\perp^2\mathbf{E}_\perp + q^2\mathbf{E}_\perp = -(iq^2\phi/k)\nabla_\perp E_z, \tag{15}$$

$$\nabla_\perp\cdot\left(c^2 + \frac{\gamma^2\omega_T^2}{\Omega^2}\right)\nabla_\perp E_z + \left(\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega^2}{\gamma^2\Omega^2}\right)E_z = -\frac{ic^2}{k}\nabla_\perp\cdot q^2\phi\mathbf{E}_\perp, \tag{16}$$

where

$$q^2c^2 \equiv -k^2c^2 + \omega^2 - \omega_P^2 - \omega_T^2, \tag{17}$$

$$q^2c^2\phi \equiv -k^2c^2 + (kv\omega_T^2/\Omega). \tag{18}$$

We have not found it possible to find any transformation that decouples these equations in the general case. However, progress is possible in the special cases with which we shall be concerned. We define a "transverse field" $\mathbf{T}$ by

$$\mathbf{T} \equiv \mathbf{E}_\perp + (i/k)\phi\nabla_\perp E_z \tag{19}$$

and rewrite (15) and (16) as

$$-\nabla_\perp(\nabla_\perp\cdot\mathbf{T}) + \nabla_\perp^2\mathbf{T} + q^2\mathbf{T} = (-i/k)[\nabla_\perp(\nabla_\perp\cdot\phi\nabla_\perp E_z) - \nabla_\perp^2(\phi\nabla_\perp E_z)], \tag{20}$$

$$\nabla_\perp\cdot\frac{1}{q^2}\left[\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega^2}{\Omega^2} - \frac{\omega_T^2(\omega^2 - \omega_P^2)}{\gamma^2\Omega^2}\right]\nabla_\perp E_z + \left(\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega^2}{\gamma^2\Omega^2}\right)E_z = -\frac{iq^2c^2}{k}\mathbf{T}\cdot\nabla_\perp\phi, \tag{21}$$

and we see from (20) that $\mathbf{T}$ is transverse, in the sense that

$$\nabla\cdot(q^2\mathbf{T}) = 0; \qquad T_z = 0. \tag{22}$$

At great distances from the beam we can set $\omega_T^2 = 0$, so $q^2c^2 = -k^2c^2 + \omega^2 - \omega_P^2$ and $\phi = -k^2/q^2$ are constant. Equations (20) and (21) then read

$$\nabla_\perp^2\mathbf{T} + q^2\mathbf{T} = 0; \qquad \nabla\cdot\mathbf{T} = 0, \tag{23}$$

$$\nabla_\perp^2 E_z + q^2E_z = 0, \tag{24}$$

so $\mathbf{T}$ and E_z will be linear combinations of solutions with space dependence in cylindrical coordinates $e^{ikz}e^{im\theta}K_m(\pm iqr)$. Since $|\omega^2 - \omega_P^2| \ll k^2c^2$ (see below) the imaginary part of q will certainly be appreciable (of order k). The acceptable solutions then decay as $r \to \infty$ like $\exp(-|\mathrm{Im}\, q|\, r)$. This corresponds physically to the fact that a plasma will only transmit disturbances for which $\omega^2 = \omega_P^2$ exactly [so that (21) with $\omega_T^2 = 0$ is satisfied trivially and does not lead to (24)] or for which $\omega^2 - \omega_P^2 - k^2c^2 = c^2k_\perp^2 > 0$, i.e., $k^2c^2 < \omega^2 - \omega_P^2$. Hence no energy may leave the beam. This would not be the case if ω_P^2 decreased at great distances below $\omega^2 - k^2c^2$; then the beam could lose energy at a rate governed by a Gamow penetration factor.

It may be noted that if ϕ is constant everywhere then E_z and $\mathbf{T}$ are completely uncoupled, since the right-hand sides of (20) and (21) vanish. This point will be pursued in Sec. VII. [Also, if the beam and plasma have circular symmetry about the z axis, and the fields are θ independent, then since $q^2\mathbf{T}$ is solenoidal it must be in the θ direction, and hence perpendicular to $\nabla_\perp\phi$, so that again the right-hand side of (21) must vanish.]

III. WEAK-BEAM ASSUMPTION

If ω_T^2 is sufficiently small, it is certainly reasonable that there should exist a mode which is very close to a simple longitudinal electrostatic plasma wave. In examining this mode, we shall make the defining assumptions:

$$\omega_T^2 \ll |\omega^2 - \omega_P^2| \ll |\omega_P^2|, \tag{25}$$

$$\omega_T^2 \ll |\Omega|^2 \ll |\omega_P^2|, \tag{26}$$

so that ω_P, kv, and ω are all roughly equal. The motivation for (26) arises from our experience with the case of an infinite beam; we know there that (26) is satisfied by the most rapidly growing modes. We shall furthermore assume that the variations in ω_P^2 are at most of the same magnitudes as those in ω_T^2.

It follows then that $q^2 \simeq -k^2$, and that the variations in ϕ are of order $\beta\omega_T^2/\Omega\omega_P$. Since the right-hand side of (20) would vanish for ϕ constant, there exist solutions with $\mathbf{T}$ of order $\beta\omega_T^2/\Omega\omega_P$ with respect to E_z. For such solutions the right-hand side of (21) is smaller then the left-hand side by a factor

$$k^2c^2\left(\frac{\beta\omega_T^2}{\Omega\omega_P}\right)^2(\omega^2 - \omega_P^2)^{-1} \simeq \frac{\omega_T^4}{\Omega^2(\omega^2 - \omega_P^2)} \ll 1$$

or

$$k^2c^2\left(\frac{\beta\omega_T^2}{\Omega\omega_P}\right)^2\left(\frac{\omega_T^2\omega_P^2}{\Omega^2}\right)^{-1} = \frac{k^2v^2\omega_T^2}{\omega_P^4} \simeq \frac{\omega_T^2}{\omega_P^2} \ll 1.$$

Dropping the right-hand side of (21), and higher-order terms on the left side, we have

$$0 = \nabla_\perp\cdot\left(\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega_P^2}{\Omega^2}\right)\nabla_\perp E_z - k^2\left(\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega_P^2}{\gamma^2\Omega^2}\right)E_z. \tag{27}$$

The variations in $\omega^2 - \omega_P^2$ will be much smaller (by the factor ω_P^2/Ω^2) than those in $\omega_T^2\omega_P^2/\Omega^2$, so we may divide by $\omega^2 - \omega_P^2$ as if it were constant, and obtain finally

$$\nabla_\perp\cdot[1 - \eta\,\Delta(\mathbf{r})]\nabla_\perp E_z - k^2[1 - \eta\,\Delta(\mathbf{r})/\gamma^2]E_z = 0, \tag{28}$$

where

$$\eta \equiv \omega_{T0}^2\omega_P^2/\Omega^2(\omega^2 - \omega_P^2), \tag{29}$$

$$\Delta(\mathbf{r}) \equiv \omega_T^2(\mathbf{r})/\omega_{T0}^2. \tag{30}$$

It is convenient to take ω_{T0}^2 as the maximum value of ω_T^2, so that the beam shape function Δ is positive and varies from zero at infinity to unity on the beam axis. For example, the Bennett distribution corresponds to

$$\Delta = [1 + (r^2/R^2)]^{-2}, \tag{31}$$

where

$$R = (\omega_R^2 m_e\beta^2\gamma^2/8kT)^{-\frac{1}{2}} \tag{32}$$

(here T is the beam temperature and k is Boltzmann's constant). Equation (28) is the fundamental equation to be discussed in succeeding sections.

[For completeness, it should be mentioned that for $\omega_T^2 \ll \omega_P^2$, there will of course also be a mode which is very close to a transverse electromagnetic plasma wave. For this mode $E_z \ll |\mathbf{T}|$,

$$\omega^2 - \omega_P^2 - k^2c^2 \gg \omega_T^2,$$

and

$$-\nabla_\perp(\nabla_\perp\cdot\mathbf{T}) + \nabla_\perp^2\mathbf{T} + q^2\mathbf{T} = 0.$$

Since

$$q^2c^2 \simeq -k^2c^2 + \omega^2 - \omega_P^2$$

is almost constant, this gives

$$(\nabla_\perp^2 + q^2)\mathbf{T} = 0, \qquad \nabla_\perp\cdot\mathbf{T} = 0$$

with solutions

$$T_r = (-im/r)e^{im\theta}\mathrm{J}_m(qr), \qquad T_\theta = qe^{im\theta}\mathrm{J}_m'(qr).$$

Since $q^2 > 0$, these solutions oscillate as $r \to \infty$, representing the actual radiation of energy by the beam.]

IV. SINGULARITIES OF INSTABILITIES

It is a remarkable feature of this problem that Eq. (28), as it stands, has *no* solutions that are free of singularities. To show this, we will start by proving that such solutions could only exist for

$$\eta \text{ real}, \qquad \eta \geq 1. \tag{33}$$

Multiplying (28) by E_z^* and integrating by parts gives

$$0 = \int dx\, dy \left[(1 - \eta \Delta) |\nabla_\perp E_z|^2 + k^2\left(1 - \frac{\eta \Delta}{\gamma^2}\right) |E_z|^2 \right]. \quad (34)$$

(The surface terms give no contribution for, as we have seen, the regular solution decays exponentially as $r \to \infty$.) Taking the imaginary part of (34) yields

$$0 = \mathrm{Im}\,(\eta) \int dx\, dy \left(|\nabla_\perp E_z|^2 + \frac{k^2}{\gamma^2} |E_z|^2 \right) \Delta, \quad (35)$$

so η must be real. Since the integrand of (34) must change sign somewhere, we must also have $\eta \geq 1$.

It is now apparent that the "allowed" values of η are just those for which the coefficient $(1 - \eta \Delta)$ of the second derivative in (28) vanishes at some point r_0. For example, using (31) gives

$$r_0(\eta) = R(\eta^{\frac{1}{2}} - 1)^{\frac{1}{2}}.$$

But at such points the differential equation has a regular singularity. If we assume that $\Delta = \Delta(r)$ is θ independent and write in cylindrical coordinates

$$E_z = e^{ikz} e^{im\theta} F(r) e^{-i\omega t}, \quad (36)$$

then (28) becomes

$$0 = \frac{1}{r}\frac{d}{dr} r[1 - \eta \Delta(r)] \frac{dF}{dr} - \frac{m^2}{r^2}[1 - \eta \Delta(r)]F - k^2\left[1 - \frac{\eta \Delta(r)}{\gamma^2}\right]F \quad (37)$$

[we may choose $F(r)$ to be real].
Near r_0 the two independent solutions $F^{(1)}$ and $F^{(2)}$ behave like

$$F^{(1)}(r) \sim f(r - r_0), \quad (38)$$

$$F^{(2)}(r) \sim f(r - r_0) \ln (r - r_0) + g(r - r_0), \quad (39)$$

where f and g are given by power series with $f(0) \neq 0$. If we impose the condition that $F = F^{(1)}$ to avoid the logarithm at $r = r_0$, then in general, for any η, F will be singular when continued to $r = 0$ or $r = \infty$.

In fact, we can make a stronger statement. For any $\eta > 1$, $F(r)$ *must* be singular at $r = r_0$ or at infinity. This may be seen by converting (37) into a Riccati equation,

$$\frac{d}{dr}(1 - \eta \Delta)\psi = (1 - \eta \Delta) \cdot \left(\frac{m^2}{r^2} - \frac{\psi}{r} - \psi^2 + k^2\right) + \beta^2 k^2 \eta \Delta, \quad (40)$$

where

$$\psi(r) = (d/dr) \ln F(r). \quad (41)$$

Clearly, if $F(r)$ is regular at infinity then as $r \to \infty$, $\psi(r) \to -k < 0$. The only way $\psi(r)$ can become positive for finite r is if it vanishes at some r_1. [Note that even if $\psi(r)$ can change sign by becoming infinite at $r = r_2$, then $F(r_2) = 0$; since $F(r) \sim e^{-kr}$ as $r \to \infty$, there must be a finite $r_1 > r_2$ at which $F'(r_1) = 0$, and hence $\psi(r_1) = 0$.] But at any r_1 for which ψ vanishes

$$\left.\frac{d\psi}{dr}\right|_{r_1} = \frac{m^2}{r_1^2} + k^2\left\{\frac{1 - [\eta \Delta(r_1)/\gamma^2]}{1 - \eta \Delta(r_1)}\right\}, \quad (42)$$

and this is positive if $r_1 > r_0$. Hence $\psi(r)$ cannot *decrease* through zero for $r > r_0$, and since it is negative at infinity it must be negative for all $r > r_0$. On the other hand, if $F(r)$ is regular at r_0 then (40) gives

$$\psi(r_0) = -\beta^2 k^2 \Delta(r_0)/\Delta'(r_0), \quad (43)$$

which is *positive* since Δ is decreasing at r_0. Hence if $F(r)$ is regular at infinity it cannot be regular at r_0.

Incidentally, if $\eta < 1$, then the above argument shows that $\psi(r) < 0$ for all r. But if $F(r)$ is regular at $r = 0$ then as $r \to 0$, $F(r) \sim r^m$, so $\psi(0) = m \geq 0$. This serves as an alternate proof that there are no solutions with $\eta < 1$ that are regular at $r = 0$ and $r = \infty$.

We are now faced with an unavoidable singularity in $F(r)$; it is at $r = 0$ or $r = \infty$ except for real $\eta \geq 1$, in which case it is at $r = r_0$, if we choose $F(r)$ to be regular at $r = 0$ and $r = \infty$. Of course, no physical instability can have fields which are infinite anywhere, and so it might be thought that the two-stream instability does not exist for a finite beam. We do not accept this conclusion. The two-stream instability is known to exist for an infinite beam, and hence cannot be entirely absent for one that is large but finite.

To resolve this paradox, we must recall that the electric field in a particular instability is not given by a single term

$$e^{-i\omega t} e^{ikz} e^{im\theta} F_m(r, k, \omega), \quad (44)$$

where F_m satisfies (25), but by an integral of such terms,

$$E_z = \sum_m \int d\omega\, dk\, e^{-i\omega t} e^{ikz} e^{im\theta} F_m(r, k, \omega). \quad (45)$$

Here the normalization of the F_m for different m, k, and ω, is determined by the initial conditions characterizing the disturbance that triggers the instability (see Appendix). If F_m has a singularity at any fixed point, such as $r = 0$ or $r = \infty$, then so will E_z, and such solutions are unphysical. On the other hand, if F_m has a singularity at a location that depends on ω or k, such as $r = r_0(\eta)$, and if the singularity is mild enough for the integral (45) to converge, as in the case of the logarithmic term in (39), then the field E_z is well defined, and singular nowhere, the integration having smeared out the singularity.

We are thus led to the conclusion that the *two-stream instability can exist, but only if η is real and* $\eta \geq 1$. The remaining question is whether additional conditions on E_z can "quantize" these values of η. In Sec. V we shall show that in general η is not quantized; however in Sec. VI we show that for a sufficiently sharp beam edge the spectrum of η values is effectively discrete.

V. SPECTRUM OF INSTABILITIES

In order to interpret the condition $\eta \geq 1$, it is convenient to consider the case of an infinite beam with constant $\omega_B^2 = \omega_{B0}^2$. In this case (28) may be solved by setting $E_z \sim \exp\,(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp)$; we obtain

$$\eta = (1 - \beta^2 \cos^2 \theta)^{-1} \tag{46}$$

where

$$\theta = \tan^{-1}\,(|k_\perp|/k) \tag{47}$$

is the angle between the direction of propagation and the beam velocity.

We see that the spectrum of allowed η for the infinite beam is given by the condition

$$1 \leq \eta \leq \gamma^2. \tag{48}$$

This range of η values certainly satisfies the general condition $\eta \geq 1$. As we shall see, the additional restriction $\eta \leq \gamma^2$ does not affect the most rapidly growing modes.

The problem of expressing the frequency ω in terms of η has been solved for the infinite uniform beam[3]; we shall essentially just go through the same steps here for the reader's convenience. Equation (26) may be written

$$(\omega^2 - \omega_P^2)(\omega - kv)^2 = \omega_T^2\omega_P^2/\eta. \tag{49}$$

This is actually a quartic equation in ω, since the dependence of ω_P^2 on ω is given by

$$\omega_P^2 = \omega_R^2\omega/(\omega + i\omega_c), \tag{50}$$

where $\omega_R^2 = 4\pi n_P e^2/m_e$ is the real plasma frequency for zero collision frequency ω_c. Using (50), (49) becomes

$$[\omega(\omega + i\omega_c) - \omega_R^2](\omega - kv)^2 = 2\omega_R\epsilon^3, \tag{51}$$

where

$$\epsilon = |\omega_{T0}^2\omega_R/2\eta|^{\frac{1}{3}} > 0. \tag{52}$$

For an infinite beam, the corresponding formula is

$$\epsilon = |\tfrac{1}{2}\omega_T^2\omega_R(1 - \beta^2 \cos^2 \theta)|^{\frac{1}{3}}.$$

Now, we are looking for modes with $\omega^2 \simeq \omega_P^2$. We shall also assume from now on that $\omega_c \ll \omega_R$. From (50), we have then,

$$\omega \simeq \omega_0 \equiv \omega_R - \tfrac{1}{2}i\omega_c. \tag{53}$$

Hence (51) may be reduced to a cubic,

$$(\omega - \omega_0)(\omega - kv)^2 = \epsilon^3. \tag{54}$$

The roots are

$$\omega = \omega_0 - 2\,\delta + \chi + (\delta^2/\chi), \tag{55}$$

or in other words

$$\Omega = \delta + \chi + (\delta^2/\chi), \tag{56}$$

where

$$\delta \equiv \tfrac{1}{3}(\omega_0 - kv) = \tfrac{1}{3}(\omega_R - \tfrac{1}{2}i\omega_c - kv) \tag{57}$$

and χ runs over the three cube roots,

$$\chi = \{\tfrac{1}{2}\epsilon^3 + \delta^3 + [\epsilon^3(\tfrac{1}{4}\epsilon^3 + \delta^3)]^{\frac{1}{2}}\}^{\frac{1}{3}}. \tag{58}$$

Evidently, there are two special cases of interest.

I. $|\delta| \ll \epsilon$ (i.e., $|\omega_R - kv| \ll \epsilon$ and $\omega_c \ll \epsilon$).

In this case the three roots are

$$\omega = \omega_0 + \mathfrak{z}\epsilon - 2\,\delta + O(\delta^2/\epsilon), \tag{59}$$

where $\mathfrak{z}^3 = 1$, and the maximum growth rate (since $\omega_c \ll \epsilon$) is

$$(\text{Im}\,\omega)_{\max} \simeq \tfrac{1}{2}\sqrt{3}\,\epsilon. \tag{60}$$

II. $|\delta| \gg \epsilon$ (i.e., $|\omega_R - kv| \gg \epsilon$ *or* $\omega_c \gg \epsilon$).

In this case the three roots are

$$\omega = \omega_0 + (\mathfrak{z} + \mathfrak{z}^* - 2)\,\delta + \tfrac{1}{3}(\epsilon^3/\delta)^{\frac{1}{2}}(\mathfrak{z} - \mathfrak{z}^*) + O(\epsilon^3/\delta^2), \tag{61}$$

where $\mathfrak{z}^3 = 1$; in other words, the roots are

$$\omega = \omega_0 + O(\epsilon^3/\delta^2) \tag{62}$$

with negative growth rate $\text{Im}\,\omega \simeq -\frac{1}{2}\omega_c$, and

$$\omega = kv \pm i(\epsilon^3/3\,\delta)^{\frac{1}{2}} + O(\epsilon^3/\delta^2). \tag{63}$$

For the latter roots it is necessary to distinguish between three subcases:

IIA. $\omega_c \gg |\omega_R - kv|$ (so $\omega_c \gg \epsilon$).

Here $3\delta \simeq -\frac{1}{2}i\,\omega_c$, so

$$\omega = kv \pm (1 - i)(\epsilon^3/\omega_c)^{\frac{1}{2}} + O(\epsilon^3/\delta^2) \tag{64}$$

and the maximum growth rate is

$$(\operatorname{Im}\omega)_{\max} \simeq (\epsilon^3/\omega_c)^{\frac{1}{2}}. \tag{65}$$

IIB. $\omega_c \ll \omega_R - kv$ (so $\omega_R - kv \gg \epsilon$).

Here $3\delta \simeq \omega_R - kv > 0$, so

$$\omega = kv \pm i[\epsilon^3/(\omega_R - kv)]^{\frac{1}{2}} + O(\epsilon^3/\delta^2) \tag{66}$$

and the maximum growth rate is

$$(\operatorname{Im}\omega)_{\max} = [\epsilon^3/(\omega_R - kv)]^{\frac{1}{2}}. \tag{67}$$

IIC. $\omega_c \ll kv - \omega_R$ (so $kv - \omega_R \gg \epsilon$).

Here $3\delta \simeq \omega_R - kv < 0$, so

$$\omega = kv \pm [\epsilon^3/(kv - \omega_R)]^{\frac{1}{2}} + O(\epsilon^3/\delta^3) \tag{68}$$

and the growth rate is zero, to order ϵ^3/δ^2.

The contours Γ corresponding to the condition $\eta \geq 1$ are shown in Fig. 1, for all four cases. The main result for our purposes is that the fastest growing mode has growth rate proportional to $\eta^{-\frac{1}{3}}$ (case I) or to $\eta^{-\frac{1}{2}}$ (case IIA, B).

It is now incumbent on us to check back and make sure that the values of Ω derived here satisfy condition (26), at least for the most rapidly growing modes. In case I, $|\Omega| \simeq \epsilon$, so

$$|\Omega|^2/|\omega_P|^2 \simeq |\omega_{T0}^2/2\eta\omega_R^2|^{\frac{2}{3}} \ll 1$$

and

$$|\Omega|^2/|\omega_{T0}^2| \simeq |\omega_R^2/4\eta^2\omega_{T0}^2|^{\frac{1}{3}},$$

which is large compared to one, except for the slowly growing modes with large η. In case II the mode corresponding to $\mathfrak{z} = 1$ is damped, and in fact our analysis does not work here, as $|\Omega|$ is not necessarily small compared to $|\omega_P|$ (although for low η it is large compared to ω_{T0}). For the other modes, corresponding to $\mathfrak{z} = \exp(\pm\frac{2}{3}i\pi)$, we have $|\Omega|^2 \simeq |\epsilon^3/2\delta|$ and since $|\delta| \gg \epsilon$ by definition,

$$|\Omega|^2/|\omega_P|^2 \ll \epsilon^2/2\omega_R^2 = |\omega_{T0}^2/2\eta\omega_R^2|^{\frac{2}{3}} \ll 1.$$

Also, if we look only for the fastest growing instabilities, we can assume that the beam is "tuned", i.e., $|\omega_R - kv| \lesssim \omega_c \ll \omega_R$ so that $|\delta| \ll \omega_R$ and hence

$$|\Omega|^2/\omega_{T0}^2 \gg \epsilon^3/2\omega_R\omega_{T0}^2 \simeq (4\eta)^{-1}.$$

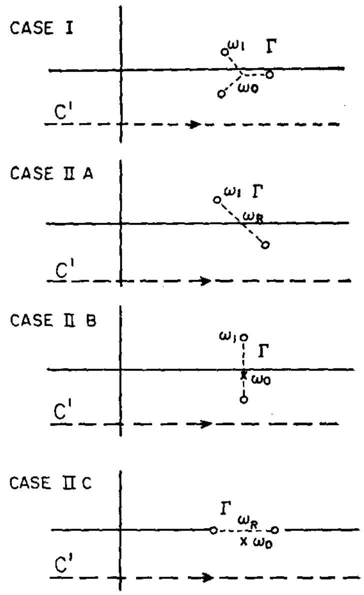

FIG. 1. The contours Γ in the complex plane of ω on which a solution exists to the fundamental differential equation (28), for the cases:

I. $|\omega_R - kv| \ll \epsilon$, $\omega_c \ll \epsilon$.
IIA. $\omega_c \gg |\omega_R - kv|$, $\omega_c \gg \epsilon$.
IIB. $\omega_R - kv \gg \omega_c$, $\omega_R - kv \gg \epsilon$.
IIC. $kv - \omega_R \gg \omega_c$, $kv - \omega_R \gg \epsilon$.

The actual electric field is given by an integral of the solution of (28) times $e^{-i\omega t}$ over the contour Γ, plus an integral over C' which may be ignored since C' can be lowered indefinitely. The behavior as $t \to \infty$ in cases I, II A, and II B is determined by the properties of the integrand near $\omega = \omega_1$. The circled points correspond to $\eta = 1$. The point $\omega_0 = \omega_R - \frac{1}{2}i\omega_c$, where $\omega_R \simeq |\omega_P|$; the cross at this point in case II indicates that part of Γ is a small segment near ω_0, corresponding to $1^{\frac{1}{3}} = 1$. The approximations made in deriving (28) are not really valid at ω_0 unless the beam is "tuned", i.e., $|\omega_R - kv| \ll \omega_R$, and are not valid along the dotted portions of Γ which correspond to large values of η.

This is much greater than one for the rapidly growing modes with small η.

If the values of η and hence ϵ are quantized, as in the case of a sharp finite beam (Sec. VI) then what we have found here are the actual growth rates of a set of discrete modes. If a continuum is allowed, as in Sec. V, then we must look more closely at the integral (45) for E_z to determine its actual behavior as $t \to \infty$.

Suppose the values of ω for which $F_m \neq 0$ lie on a contour Γ reaching a maximum height above the real axis at one end ω_1 (see Fig. 1). If F_m approached a constant as $\omega \to \omega_1$, then the behavior of E_z as $t \to \infty$ would be essentially given by $\exp(-i\omega_1 t)$. However, F_m must vanish as ω approaches ω_1 along Γ in order that $(\partial E_z/\partial r) = (-i\omega/c)B_\theta + ikE_r$ should be finite at $r = r_0(\omega_1)$. (Otherwise at this point,

$$\partial F_m/\partial r \sim [r_0(\omega) - r_0(\omega_1)]^{-1} \quad \text{as} \quad \omega \to \omega_1.$$

and the integral for $\partial E_z/\partial r$ would diverge logarithmically.) If $F_m \sim (\omega - \omega_1)^\nu$ as $\omega \to \omega_1$ along Γ, then the asymptotic behavior of E_z as $t \to \infty$ is essentially given by a factor $t^{-\nu-1} \exp(-i\omega_1 t)$.

VI. SOLUTION OF THE EQUATION

We turn now to the problem of constructing explicit solutions of Eq. (37). Our first task is to establish a connection formula for the solution across the singularity at $r = r_0$. This result is expressed by Eq. (80). We next consider the case that the radial wavelength of the instability is small compared to the transverse beam dimensions. This permits us to use the WKB approximation to exhibit an explicit solution.

Since the differential equation (37) does not have physically acceptable solutions, we are forced to construct a wave-packet solution, such as that given by Eq. (45). Let us then imagine that at time $t = 0$ an external mechanism has modified the beam in some prescribed manner. It is the development in time of this disturbance which we wish to study.

It is convenient to first rewrite Eq. (37) in the form

$$\frac{1}{y}\frac{d}{dy}\left[yP(y)\frac{dF}{dy}\right] - \frac{m^2}{y^2}P(y)F + L(y)F = 0. \tag{69}$$

Here we have introduced [see Eqs. (27) and (28)]

$$\begin{aligned} y &\equiv kr, \\ P(y) &\equiv \eta\,\Delta(y) - 1, \\ L(y) &\equiv 1 - [\eta\,\Delta(y)/\gamma^2]. \end{aligned} \tag{70}$$

Equation (69) defines solutions $F_m(r, k, \omega)$ everywhere except at the singular point $y_0 = kr_0$, these solutions being regular at $y = 0$ and decaying exponentially as $y \to \infty$.

Corresponding to a particular value of η, we may always define a "radial" wave number $\xi = k_\perp/k$ using Eqs. (46) and (47):

$$\eta = [1 - \beta^2/(1 + \xi^2)]^{-1}. \tag{71}$$

We may then consider the frequency ω to be a function of ξ:

$$\omega = \omega(\xi). \tag{72}$$

A wave-packet solution to our equation is then of this form

$$E_z(t) = \sum_m \int dk\, e^{ikz} e^{im\theta} E_z(k, m, r), \tag{73}$$

where

$$E_z(k, m, r) = \int e^{-i\omega(\xi)t} F_m(r, k, \xi) A(k, m, \xi)\, d\xi. \tag{74}$$

Here A represents the wave-packet amplitude and the F_m are some appropriate set of solutions to Eq. (69).

We require only that $E_z(t)$ be a solution to our time-dependent equation and that it reduce to our prescribed disturbance at $t = 0$. The condition that $E_z(t)$ be a solution to the time-dependent differential equations is equivalent to the condition that [see Eq. (69)]

$$0 = I(y) \equiv \int e^{-i\omega(\xi)t}\left\{\frac{1}{y}\frac{d}{dy}\left[yP(y)\frac{dF}{dy}\right] - \frac{m^2}{y^2}P(y)F + L(y)F\right\}A(\xi)\, d\xi. \tag{75}$$

(We have here dropped the labels k and m in A and F.) Since F satisfies Eq. (69) except at $y = y_0(\xi)$, we may rewrite Eq. (75) as

$$0 = I(y) = \lim_{\delta \to 0+} \int_{\xi_0(y)-\delta}^{\xi_0(y)+\delta} e^{-i\omega(\xi)t}\left\{\frac{1}{y}\frac{d}{dy}\left[yP(y)\frac{dF}{dy}\right] - \frac{m^2}{y^2}P(y)F + L(y)F\right\}A(\xi)\, d\xi, \tag{76}$$

where $\xi_0(y)$ is the solution of the equation

$$y = y_0(\xi). \tag{77}$$

Because F has only a logarithmic singularity at $y = y_0$ the last two terms in parentheses in the integrand in Eq. (76) give no contribution in the limit $\delta \to 0$. We may then replace Eq. (76) by

$$\begin{aligned} 0 &= I(y) \\ &= \lim_{\delta \to 0+} \int_{\xi_0(y)-\delta}^{\xi_0(y)+\delta} e^{-i\omega(\xi)t}\left\{\frac{1}{y}\frac{d}{dy}\left[yP(y)\frac{dF}{dy}\right]\right\}A(\xi)\, d\xi \\ &= A(\xi_0)e^{-i\omega(\xi_0)t} \\ &\quad \cdot \lim_{\delta \to 0+} \int_{\xi_0(y)-\delta}^{\xi_0(y)+\delta} \left\{\frac{1}{y}\frac{d}{dy}\left[yP(y)\frac{dF}{dy}\right]\right\} d\xi. \end{aligned} \tag{78}$$

Now in the neighborhood of $y = y_0$, we may consider $P(y)\, dF/dy$ to be a function of $y - y_0$, so

$$\frac{d}{dy}\left[P\frac{dF}{dy}\right] = -\frac{d}{dy_0}\left[P\frac{dF}{dy}\right].$$

We may also write $d\xi = (d\xi_0/dy)\, dy_0$. Thus

$$0 = I(y) = A(\xi_0)e^{-i\omega(\xi_0)t} \cdot \lim_{\delta' \to 0+} \int_{y_0-\delta'}^{y_0+\delta'} \frac{d}{dy_0}\left(P\frac{dF}{dy}\right) dy_0 \frac{d\xi}{dy},$$

or

$$0 = I(y) = A(\xi_0)\frac{d\xi_0}{dy}e^{-i\omega(\xi_0)t} \cdot \lim_{\delta'\to 0}\left[P(y)\frac{dF}{dy}\right]\Big|_{y_0=y-\delta'}^{y_0=y+\delta'}. \quad (79)$$

In order that $E_z(t)$ be a solution to our original equations, we then require that for small δ,

$$P(y)\frac{dF}{dy}\Big|_{y=y_0(\xi)-\delta} = P(y)\frac{dF}{dy}\Big|_{y=y_0(\xi)+\delta}. \quad (80)$$

Suppose then that in the interval $y < y_0(\xi)$, a solution of Eq. (69), regular at $y = 0$, is

$$F(y) \equiv F_<(y). \quad (81)$$

Similarly, for $y > y_0(\xi)$, a solution of Eq. (69) which decays exponentially as $y \to \infty$ is

$$F(y) \equiv F_>(y). \quad (82)$$

The condition (80) provides the precise relation needed to join these two solutions in order that we have a physically acceptable solution for the electric field intensity $E_z(t)$. Such a solution may be found for any $\eta \geq 1$. For each $\eta \geq 1$ we may find an ω corresponding to an unstable "mode", as has already been described in Sec. IV. The field $E_z(t)$ then represents a packet of the unstable "modes."

We shall now illustrate this by constructing a solution for the case of short wavelengths, so that we may use the WKB method. Let us assume that $\eta/\gamma^2 < 1$, so

$$L(y) \geq 0 \quad (\text{all} \quad y). \quad (83)$$

This corresponds to the solutions which grow most rapidly. We shall also assume that $d\Delta/dy = 0$ at $y = 0$. Then, near the beam axis Eq. (69) has the regular solution

$$F = NJ_m[y(L_0/P_0)^{\frac{1}{2}}], \quad (84)$$

where N is a real constant, J_m is the Bessel function of order m and

$$L_0 = L(0), \qquad P_0 = P(0). \quad (85)$$

[Since Eq. (69) has real coefficients, no loss of generality is incurred by requiring that F be real.] For large y (recall our assumption that the radial wave number is large)

$$F \to N\left(\frac{2}{\pi y}\right)^{\frac{1}{2}}\left(\frac{P_0}{L_0}\right)^{\frac{1}{4}}\cos\left[y\left(\frac{L_0}{P_0}\right)^{\frac{1}{2}} - \tfrac{1}{4}\pi(1 + 2m)\right]. \quad (86)$$

This may also be written in the form

$$F = \frac{N}{(2\pi y)^{\frac{1}{2}}}\left(\frac{P_0}{L_0}\right)^{\frac{1}{4}}\{\exp[i\Phi - \tfrac{1}{4}i\pi(1 + 2m)] + \exp[-i\Phi + \tfrac{1}{4}i\pi(1 + 2m)]\}, \quad (87)$$

where

$$\Phi = \int_0^y dy'\left[\frac{L(y')}{P(y')}\right]^{\frac{1}{2}}, \quad (88)$$

since near the beam axis Φ is just $y(L_0/P_0)^{\frac{1}{2}}$. (The reason for introducing the quantity Φ here is that it will occur below in our discussion on the WKB solution.)

Next, in the interval $y \gg 0$, $y_0 - y \gg 0$, we try a WKB solution to Eq. (69) of the form

$$f^{\pm} = \frac{1}{[yK(y)]^{\frac{1}{2}}}\exp\left[\pm i\int_0^y dy'\,K(y')\right] \cdot \left(1 \pm i\frac{m^2 - \frac{1}{4}}{2Kr} + \cdots\right). \quad (89)$$

Substitution of this into Eq. (69), if we neglect terms of relative order

$$\left[\frac{3}{4}\frac{(K')^2}{K^4} + \frac{K''}{K^3}\right], \qquad \frac{P'K'}{PK^2}, \quad \text{and} \quad \frac{P'}{PyK} \quad (90)$$

gives the relation

$$-K^2P + iKP' + L = 0. \quad (91)$$

Solving for K, we obtain

$$K(y) \simeq [L(y)/P(y)]^{\frac{1}{2}} \pm \tfrac{1}{2}i(\partial/\partial y)\ln P. \quad (92)$$

We may use Eqs. (89) and (92) to construct a solution for F, involving a linear combination of f^+ and f^-, which agrees with Eq. (87) near the beam axis. The result is

$$F = \frac{N}{(2\pi y)^{\frac{1}{2}}}\left\{\frac{(P_0)^{\frac{1}{2}}}{[L(y)P(y)]^{\frac{1}{4}}}\right\}\{\exp[i\Phi - \tfrac{1}{4}i\pi(1 + 2m)] + \exp[-i\Phi + \tfrac{1}{4}i\pi(1 + 2m)]\}. \quad (93)$$

We now have a solution of Eq. (69) which is regular at the origin and defined for $y_0 - y \gg 1$. For $y \approx y_0$, however, the WKB approximation fails. This happens because $P(y_0) = 0$, causing the "small" quantities (90) to become large.

To find F near the turning point at $y = y_0$, we make the familiar assumption that near y_0,

$$P(y) = (y - y_0)/D, \quad (94)$$

where D is a constant, and that

$$L(y) \simeq L(y_0) = \beta^2. \quad (95)$$

We require that the approximate expression (94) be valid over an interval $|y - y_0| < \Delta y$, where Δy is sufficiently large that at $y = y_0 - \Delta y$ the expressions (90) are negligible. (This is a condition imposed on the radial wave number ξ.) The term $(m^2/y^2)P(y)$ in Eq. (90) will also be neglected, which implies an upper limit on m.

Then, with the definition

$$Y \equiv DL(y_0)(y - y_0), \tag{96}$$

Eq. (69) becomes

$$(d/dY)[Y\, dF/dY] - F = 0. \tag{97}$$

Within the turning point, corresponding to $Y < 0$, this has the solution

$$F = C_1 J_0(u) + C_2 N_0(u), \qquad (Y < 0), \tag{98}$$

where

$$u = 2(-Y)^{\frac{1}{2}}, \tag{99}$$

and $N_0(u)$ is the Neumann function.[6] C_1 and C_2 are two real constants. For $Y > 0$, we have the decaying solution

$$F = C_3 K_0(w), \tag{100}$$

where K_0 is the Hankel function of imaginary argument and

$$w = 2Y^{\frac{1}{2}}. \tag{101}$$

Near $|Y| = 0$, we have

$$\begin{aligned} F &\simeq C_1 + (C_2/\pi)[2\gamma + \ln(-Y)], & Y < 0,\\ F &\simeq -C_3(\tfrac{1}{2}\ln Y + \gamma), & Y > 0, \end{aligned} \tag{102}$$

where γ is Euler's constant.

We are now ready to use the connection formula (80) to join these two solutions. We find that

$$\begin{aligned} (y - y_0)\, dF/dy &\rightarrow C_2/\pi \quad \text{as} \quad y \rightarrow y_0(-)\\ &\rightarrow -\tfrac{1}{2}C_3 \quad \text{as} \quad y \rightarrow y_0(+). \end{aligned}$$

Therefore Eq. (80) implies that

$$C_3 = -(2/\pi)C_2. \tag{103}$$

We have yet to join (98) to the WKB expression (93). For large u,

$$\begin{aligned} J_0(u) &\rightarrow (2/\pi u)^{\frac{1}{2}} \cos(u - \tfrac{1}{4}\pi),\\ N_0(u) &\rightarrow (2/\pi u)^{\frac{1}{2}} \sin(u - \tfrac{1}{4}\pi). \end{aligned} \tag{104}$$

[6] See G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1944), p. 64. This function is called Y_0 in Watson.

Therefore, for $y_0 - y \gg 0$, Eq. (98) is equivalent to

$$\begin{aligned} F = \frac{1}{(2\pi u)^{\frac{1}{2}}} [(C_1 - iC_2) \exp(-\tfrac{1}{4}i\pi + i\Phi_0 - i\Phi)\\ + (C_1 + iC_2)\exp(\tfrac{1}{4}i\pi - i\Phi_0 + i\Phi)]. \end{aligned} \tag{105}$$

Here

$$\Phi_0 = \int_0^{y_0} dy' \left[\frac{L(y')}{P(y')}\right]^{\frac{1}{2}}. \tag{106}$$

To verify Eq. (105), we note that near the turning point

$$\begin{aligned} \Phi_0 - \Phi &= \int_y^{y_0} dy' \left[\frac{L(y_0)\, D}{y_0 - y}\right]^{\frac{1}{2}}\\ &= 2[L(y_0)\, D(y_0 - y)]^{\frac{1}{2}}\\ &= u. \end{aligned}$$

On comparing Eqs. (93) and (105), we find that

$$C_1 + iC_2 = N\frac{2P_0D}{y_0}\exp\left[i\Phi_0 - \frac{i\pi}{2}(1 + m)\right]. \tag{107}$$

This equation determines the two constants C_1 and C_2 in terms of N. Finally, Eq. (103) determines the constant C_3. We have therefore explicitly constructed the functions F for the case that the radial wave number ξ is sufficiently large that the WKB method can be used.

VII. SHARP BEAM BOUNDARY

In this section we will consider the special case of ω_T^2 constant inside the beam, and dropping rapidly to zero within a narrow beam edge at $r = R$. After the last section, it may come as a surprise that we should bother with special cases, since it was shown there that the only condition on the spectrum of instabilities for any beam shape is that $\eta \geq 1$ provided we are willing to accept solutions with a "smeared-out" logarithmic singularity. However, as we consider beam shapes with sharper and sharper edges, such solutions become increasingly unacceptable. If the region within which the singularity is "smeared" becomes very small, then the ratio of the value of the field E_z within this region to the value outside becomes very large.

Eventually for sufficiently sharp boundaries we must admit that an instability can occur only if one of our approximations, such as low beam density, zero plasma temperature, small fields, etc., breaks down within the beam edge. In particular, it has been shown[7] that for finite plasma temperature the

[7] K. A. Brueckner and M. N. Rosenbluth, "Finite beam instability," Stanford Research Institute (unpublished report).

eigenfields $F_m(r)$ are singular nowhere. But of course we do not know *a priori* just which approximation breaks down first.

Although we do not know the equations satisfied by the electric field within the beam edge, there are physical arguments which tell us how to connect the fields inside and outside the beam. Care is required, however, because of the presence of surface charges and surface currents.

Even within the beam edge we can apply Gauss' law, which may be written

$$\nabla\cdot\mathbf{E} = -(4\pi i/\omega)\nabla\cdot\mathbf{J}; \qquad \mathbf{J} = \mathbf{J}^B + \mathbf{J}^P \tag{108}$$

or

$$\int_A (\nabla_\perp\cdot\mathbf{E}_\perp + ikE_z)\,dx\,dy = -\frac{4\pi i}{\omega}\int_A (\nabla_\perp\cdot\mathbf{J}_\perp + ikJ_z)\,dx\,dy, \tag{109}$$

the area of integration A lying between two small arcs, one just within and one just without the beam. If the beam edge is sharp enough, then since E_z must be finite, the left integral is just proportional to the change δE_n of the normal component of $\mathbf{E}$ as we cross the beam edge.

However, since surface currents are unavoidable, J_z will not be finite within the beam edge. It seems reasonable to suppose though that there are no *plasma* surface currents in the "lab" frame, and no *beam* surface currents in the "beam" frame. [This is certainly the case if the currents within the beam edge are given by formulas like (2) and (3).] Then J_z^P and $J_z^{B'}$ are finite. Using (6) for J_z', we then have from (109),

$$\delta E_n = -(4\pi i/\omega)\ \delta J_n^P - (4\pi i/\Omega)\ \delta J_n^P. \tag{110}$$

Now, although we have emphasized our ignorance of the current $\mathbf{J}$ within the beam edge, we can certainly use (2) and (13) to find J_n^P and J_n^B on both sides of the edge, and hence to find ΔJ_n. Then (110) becomes

$$\delta\left[\left(\frac{\omega^2 - \omega_P^2 - \omega_T^2\omega}{\Omega}\right)E_n\right] = \frac{-i\omega v}{\Omega^2}\,\delta(\omega_T^2\nabla_n E_z). \tag{111}$$

In the weak-beam approximation we may assume that E_n is given by (19) with T_n small on both sides of the beam edge, though of course not necessarily within the beam edge. This gives

$$E_n \simeq -(i/k)\phi\nabla_n E_z, \tag{112}$$

so (111) may now be written

$$\delta[(\omega^2 - \omega_P^2 - \omega_T^2\omega^2/\Omega^2)\nabla_n E_z] = 0. \tag{113}$$

This is in fact the same as the condition

$$\delta[P(y)\ dF/dy] = 0, \tag{114}$$

which was derived in Sec. VI on the assumption that the field equation (25) holds everywhere.

The other boundary condition that we need is derived in a conventional manner; using Faraday's law in the form

$$\partial E_z/\partial r = -(i\omega/c)B_\theta + ikE_r \tag{115}$$

and using the finiteness of B_θ and E_r everywhere, we have by integration

$$\delta E_z = 0. \tag{116}$$

Of course if we took the field equation (25) literally, the integral of $\partial E_z/\partial r$ through the beam edge would not be zero in the limit of narrow beam edge.

We will show later in this section that the other boundary conditions that can be derived by similar arguments only confirm that the change of $\mathbf{T}$ across the beam edge is small.

With constant $\Delta = 1$ inside the beam, the solution of (37) is

$$F(r) \sim J_m(\xi kr), \tag{117}$$

where

$$\xi^2 = \frac{1 - (\eta/\gamma^2)}{\eta - 1}. \tag{118}$$

(For an infinite beam, $\xi^2 = |k_\perp|^2/k^2$.) Outside the beam, $\Delta = 0$, and the solution of (37) is

$$F(r) \sim K_m(kr).$$

Combining (114) and (116) we have

$$\delta[(1 - \eta\ \Delta)(\partial E_z/\partial r)(E_z)^{-1}] = 0, \tag{119}$$

and therefore

$$\xi(1 - \eta)\frac{J_m'(\xi\rho)}{J_m(\xi\rho)} = \frac{K_m'(\rho)}{K_m(\rho)}, \tag{120}$$

where $\rho = kR$, R being the beam radius.

In analyzing this dispersion relation, it is convenient to use the spectral representation for the logarithmic derivative of J_m based on the infinite product expansion[8] of $J_m(z^{\frac{1}{2}})$:

$$L_m(z) \equiv \frac{z^{\frac{1}{2}}J_m'(z^{\frac{1}{2}})}{J_m(z^{\frac{1}{2}})} = m - 2z\sum_{n=1}^{\infty}\frac{1}{j_{mn}^2 - z}, \tag{121}$$

[8] Reference 6, p. 498.

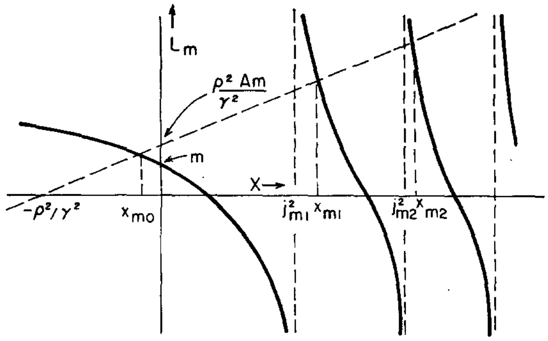

FIG. 2. A graphical representation of the solution of Eq. (125) for the eigenvalues in the case of a sharp edged beam. The parameter X is defined as $k^2R^2[1-(\eta/\gamma^2)]/(\eta-1)$; if we define $\mathbf{k}_\perp$ by (46), (47) then X reduces to $k_\perp{}^2R^2$. The abscissas X_{m0}, X_{m1}, $\cdots$ of the intersection points are the discrete allowed values of X for radial quantum number m. Although no attempt was made at accuracy in this drawing, the $j_{mn}{}^2$ here are roughly correct for $m = 1$.

where j_{mn} is the nth root of $J_m(X)$. For $n \to \infty$

$$j_{mn} = (n + \tfrac{1}{2}m - \tfrac{1}{4})\pi + O(1/n). \tag{122}$$

For low n and m,

$$\begin{aligned} j_{01} &= 2.405, \quad j_{02} = 5.520, \cdots, \\ j_{11} &= 3.832, \quad j_{12} = 7.016, \cdots. \end{aligned} \tag{123}$$

If we define $X = \rho^2\xi^2$, and note that

$$1 - \eta = \frac{-\beta^2}{1-\beta^2+\xi^2} = -\frac{\beta^2}{1-\beta^2+(X/\rho^2)} \tag{124}$$

we may rewrite (120) as

$$L_m(X) = [X + \rho^2(1-\beta^2)]A_m(\rho), \tag{125}$$

$$A_m(\rho) \equiv -\frac{K_m'(\rho)}{\beta^2\rho K_m(\rho)} \geq 0. \tag{126}$$

The first thing worth noting is that (125) can have no solutions with complex X. For,

$$\operatorname{Im} L_m(z) = -2(\operatorname{Im} z)\sum_n \frac{j_{mn}^2}{|j_{mn}^2 - z|^2} \tag{127}$$

while

$$\operatorname{Im}\{[z + \rho^2(1-\beta^2)]A_m(\rho)\} = (\operatorname{Im} z)A_m(\rho), \tag{128}$$

and the two coefficients of $\operatorname{Im} z$ are of opposite sign.

In order to find the real solutions we may read off the qualitative behavior of $L_m(X)$ from (121). For every n, $L_m(X)$ drops from $+\infty$ at $j_{mn}^2 + 0$ to $-\infty$ at $j_{mn}^2 - 0$. At all points, $L_m'(X) < 0$. For $X < j_{m1}^2$, $L_m(X)$ drops from $(-X)^{\frac{1}{2}}$ at $X \sim -\infty$ to m at $X = 0$ and $-\infty$ at $j_{m1}^2 - 0$ (see Fig. 2).

Since the right-hand side of (125) is linear with positive slope, there will be precisely one solution $X_{mn}(\rho)$ for each n, satisfying

$$j_{mn}^2 < X_{mn}(\rho) < j_{m,n+1}^2. \tag{129}$$

For large n, the X_{mn} are close to j_{mn}^2. In addition, there will be a solution X_{m0} satisfying

$$-\rho^2/\gamma^2 \leq X_{m0}(\rho) \leq j_{m0}^2 \tag{130}$$

(see Fig. 2). Using (124), these solutions correspond to

$$\eta_{mn} = 1 + \frac{\beta^2}{1-\beta^2+[X_{mn}(\rho)/\rho^2]}. \tag{131}$$

These values of η_{mn} fall in the range $1 \leq \eta \leq \gamma^2$ for $n \neq 0$, just as for a finite beam. The restriction $\eta \leq \gamma^2$ of course affects only the more slowly growing modes. For all n, $\eta \geq 1$.

For any m, β, ρ these values of η become for sufficiently large n,

$$\eta_{mn} \simeq 1 + (\beta\rho/n\pi)^2$$

so there are an infinite number of modes arbitrarily close to the case of maximum growth rate, $\eta = 1$.

On the other hand, if we consider a definite mode with fixed n, m, then as $\rho \to \infty$

$$A_m(\rho) \to 1/\beta^2\rho,$$

so (125) becomes

$$L_m(X) \simeq \rho/\beta^2\gamma^2$$

with solutions (for $n \neq 0$):

$$X_{mn}(\rho) \to j_{mn}^2, \qquad \eta_{mn} \to \gamma^2.$$

For $n = 0$ the solution is

$$X_{m0}(\rho) \to -\rho^2/\gamma^4, \qquad \eta_{m0} \to 1 + \gamma^2.$$

As $\rho \to 0$ for fixed n, m,

$$A_m(\rho) \to \frac{m}{\rho^2\beta^2}\,(m \neq 0), \qquad A_0(\rho) \to 1/\rho^2\beta^2\,|\ln\rho|,$$

so (125) becomes

$$L_m(X) \simeq mX/\rho^2\beta^2\ (m \neq 0),$$

$$L_0(X) \simeq X/\rho^2\beta^2\,|\ln\rho|,$$

with solutions (for $n \neq 0$):

$$X_{mn}(\rho) \to j_{mn}^2, \qquad \eta_{mn} \to 1.$$

For $n = 0$ the solution is, for $m \neq 0$

$$X_{m0}(\rho) \to (2\beta^2 - 1)\rho^2, \qquad \eta_{m0} \to 2,$$

and for $m = 0$.

$$X_{00}(\rho) \to -\rho^2/\gamma^2, \qquad \eta_{00} \to \infty .$$

Comparing the results we see that for $n \neq 0$ or $m \neq 0$, the maximum growth rate (which, as we saw in Sec. IV, is proportional to $\eta^{-\frac{1}{3}}$ or $\eta^{-\frac{1}{2}}$) is *less* for very wide then for very narrow beams. The opposite is true for $n = 0$, $m = 0$ this one mode becoming stable in the limit as $\rho \to 0$.

It may be of some interest to see what the dispersion relation (120) looks like if we do not make the "weak-beam" approximation. It is necessary to derive some further conditions at the beam edge, in addition to (111) and (116).

If we integrate the z component of Eq. (1) over the same area A as in (108), and use the finiteness of E_z, we find for small A

$$-c^2 \int_A (\nabla \times \nabla \times \mathbf{E})_z \, dx \, dy = -4\pi i\omega \int_A (J_z^B + J_z^P) \, dx \, dy.$$

Using Stokes' theorem and the finiteness of J_z^P and $J_z^{B'}$, we have for the change of the tangential component of $\nabla \times \mathbf{E}$ across the beam edge,

$$-c^2 \, \delta(\nabla \times \mathbf{E})_t' = -(4\pi\omega v/\Omega) \, \delta J_n^B. \qquad (132)$$

The derivation of the other boundary conditions follows familiar lines. From (1) and the finiteness of E_t, J_t^B, J_t^P, we obtain

$$\delta(\nabla \times \mathbf{E})_z = 0. \qquad (133)$$

From Faraday's law and the finiteness of $\mathbf{B}$ we get

$$\delta E_t = 0 \qquad (134)$$

and

$$\delta(\nabla \times \mathbf{E})_n = 0 \qquad (135)$$

as well as (116).

If we use (13) to give δJ_n^B in (132), and rewrite (111), (116), (132)–(135) in cylindrical coordinates, we have now

$$\delta\left[\left(\omega^2 - \omega_P^2 - \frac{\omega_T^2\omega}{\Omega}\right)E_r\right] = \frac{-i\omega v}{\Omega^2} \, \delta\left[\omega_T^2 \frac{\partial E_z}{\partial r}\right], \qquad (111a)$$

$$\delta E_z = 0, \qquad (116)$$

$$\delta\left[\left(\frac{v\omega_T^2}{\Omega} - kc^2\right)E_r\right] = -c^2 \, \delta\left[\left(\frac{\beta^2\omega_T^2}{\Omega^2} + 1\right)\frac{\partial E_z}{\partial r}\right], \qquad (132a)$$

$$\delta[(\partial/\partial r)(rE_\theta)] = im \, \delta E_r, \qquad (133a)$$

$$\delta E_\theta = 0. \qquad (134a)$$

[Condition (135) is now redundant, following from (116) and (134).]

These conditions can be more usefully written in terms of E_z and the field T given by (19). This gives at the beam edge

$$-(i/k) \, \delta(U \, \partial E_z/\partial r) = \delta(VT_r), \qquad (111b)$$

$$\delta(q^2 T_r) = 0, \qquad (132b)$$

$$\delta\left(\frac{\partial}{\partial r} rT_\theta\right) = im \, \delta T_r, \qquad (133b)$$

$$\delta T_\theta = -(m/kR)E_z \, \delta\phi, \qquad (134b)$$

where

$$Uq^2c^2 \equiv \omega^2 - \omega_P^2 - \frac{\omega_T^2\omega^2}{\gamma^2\Omega^2} - \frac{\omega_T^2\omega_P^2\beta^2}{\Omega^2}, \qquad (136)$$

$$Vk^2c^2 \equiv \omega^2 - \omega_P^2 - \omega_T^2\omega/\Omega, \qquad (137)$$

$$q^2c^2 = \omega^2 - \omega_P^2 - \omega_T^2 - k^2c^2, \qquad (17)$$

$$\phi q^2 c^2 = -k^2c^2 + kv\omega_T^2/\Omega. \qquad (18)$$

From (20)–(22) we see that when ω_T^2 and ω_P^2 are constant, the equations for $\mathbf{T}$ and E_z are

$$(\nabla_\perp^2 + q^2)\mathbf{T} = 0; \qquad \nabla \cdot \mathbf{T} = 0, \qquad (138)$$

$$(\nabla_\perp^2 + p^2)E_z = 0, \qquad (139)$$

where

$$Up^2c^2 \equiv \omega^2 - \omega_P^2 - \omega_T^2\omega^2/\gamma^2\Omega^2. \qquad (140)$$

The solution of these equations inside the beam is (omitting factors $e^{im\theta}e^{ikz}e^{-i\omega t}$),

$$E_z = AJ_m(p_i r), \qquad (141)$$

$$Tr = -(im/r)BJ_m(q_i r), \qquad (142)$$

$$T_\theta = Bq_i J_m'(q_i r), \qquad (143)$$

and the solution outside (choosing Im $p_o > 0$, Im $q_o > 0$) is:

$$E_z = CH_m^{(1)}(p_o r), \qquad (144)$$

$$T_r = -(im/r) \, DH_m^{(1)}(q_o r), \qquad (145)$$

$$T_\theta = Dq_0 H_m^{(1)}(q_o r). \qquad (146)$$

The subscripts i and o refer to values inside and outside. The boundary conditions now read

$$AJ_m(p_i R) = CH_m^{(1)}(p_o R), \qquad (116)$$

$$AU_ip_iJ_m'(p_iR) - CU_op_oH_m^{(1)\prime}(p_oR)$$
$$= (mk/R)[BV_iJ_m(q_iR) - DV_oH_m^{(1)}(q_oR)], \quad (111c)$$
$$mq_i^2BJ_m(q_iR) = mq_o^2\, DH_m^{(1)}(q_iR), \quad (132c)$$
$$Bq_iJ_m'(q_iR) - Dq_oH_m^{(1)\prime}(q_iR)$$
$$= -(m/kR)AJ_m(p_iR)\ \delta\phi. \quad (134b)$$

[Condition (133b) is now redundant, as it follows from (132b).]

We now have four homogeneous equations for the four constants A, B, C, D; their determinant gives the dispersion law,

$$\left[\frac{U_ip_iJ_m'(p_iR)}{J_m(p_iR)} - \frac{U_op_oH_m^{(1)\prime}(p_oR)}{H_m^{(1)}(p_oR)}\right] \cdot\left[\frac{J_m'(q_iR)}{q_iJ_m(q_iR)} - \frac{H_m^{(1)\prime}(q_oR)}{q_oH_m^{(1)}(q_oR)}\right] = \frac{m^2}{k^2p^2}(\delta\phi)^2. \quad (147)$$

It is possible to show by direct calculation that this "exact" dispersion relation does go over into the simple one (120) in the "weak-beam" limit.

APPENDIX. GENERAL LINEAR-STABILITY THEORY

The discussion in Sec. V made use of some general features of the linear theory of instabilities. We will go into them here in more detail.

Suppose a small perturbation of some system can be described by a field (or set of fields) $E(\mathbf{r}, t)$. Generally E satisfies for $t > 0$ an unforced differential equation.

$$D[-i\nabla, i(\partial/\partial t)]E(\mathbf{r}, t) = 0, \quad (148)$$

where D is a time-independent but generally $\mathbf{r}$-dependent differential operator (or matrix of operators.) As usual, we can define a Laplace transform

$$E(\mathbf{r}, \omega) = \int_0^\infty e^{i\omega t}E(\mathbf{r}, t)\, dt \quad (149)$$

for all values of ω with Im ω greater than some limiting value a. It obeys an equation of the form

$$D(-i\nabla, \omega)E(\mathbf{r}, \omega) = S(\mathbf{r}, \omega), \quad (150)$$

where S depends linearly on the value and derivatives of E at $t = 0$.

For an infinite uniform system the treatment of (150) is entirely familiar. For any physically possible perturbation we may write

$$S(\mathbf{r}, \omega) = \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}S(\mathbf{k}, \omega), \quad (151)$$

and the solution of (150) is

$$E(\mathbf{r}, \omega) = \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, D^{-1}(\mathbf{k}, \omega)S(\mathbf{k}, \omega). \quad (152)$$

Inverting the Laplace transform then gives

$$E(\mathbf{r}, t) = \frac{1}{2\pi}\int_C d\omega \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\omega t}\, D^{-1}(\mathbf{k}, \omega)S(\mathbf{k}, \omega), \quad (153)$$

the contour C running horizontally from $\omega = -\infty + ia$ to $\omega = \infty + ia$. If we deform C by lowering it sufficiently, then

$$E(\mathbf{r}, t) = \sum_i \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}R_i(\mathbf{k})e^{-i\omega_i(\mathbf{k})t}, \quad (154)$$

where the $\omega_i(\mathbf{k})$ are the zeros of $D(\mathbf{k}, \omega)$ [or det $D(\mathbf{k}, \omega)$] for a given $\mathbf{k}$, and $R_i(\mathbf{k})$ is the residue of $D^{-1}(\mathbf{k}, \omega)S(\mathbf{k}, \omega)$ at the pole $\omega = \omega_i$. Hence we see that E is a discrete linear combination of eigensolutions of (150) with $S = 0$; the coefficients are given by the initial conditions at $t = 0$.

Things are somewhat more complicated for a nonuniform system, where D depends on $\mathbf{r}$ and we cannot use (152). The inversion of the Laplace transform (149) gives

$$E(\mathbf{r}, t) = \frac{1}{2\pi}\int_C d\omega\, E(\mathbf{r}, \omega)e^{-i\omega t}, \quad (155)$$

where again C runs from $\omega = -\infty + ia$ to $\omega = \infty + ia$. As it stands, this formula tells us nothing about the asymptotic behavior of $E(\mathbf{r}, t)$ for large t. We must lower the contour of integration C at least below the real axis, obtaining from Cauchy's theorem

$$E(\mathbf{r}, t) = \frac{1}{2\pi}\int_{C'} d\omega\, E(\mathbf{r}, \omega)e^{-i\omega t} + \int_\Gamma \rho(\mathbf{r}, \omega)e^{-i\omega t}\, d\omega + \sum_n \rho_n(\mathbf{r})e^{-i\omega_n t}, \quad (156)$$

where C' runs horizontally from $\omega = -\infty - ib$ to $\omega = +\infty - ib$ $(b > 0)$, Γ consists of all parts of branch lines of $E(\mathbf{r}, \omega)$ lying between C and C', and the ω_n are the poles of $E(\mathbf{r}, \omega)$ between C and C' (see Fig. 1). The functions $i\rho_n(\mathbf{r})$ are the residues at these poles, and

$$i\rho(\mathbf{r}, \omega) = E(\mathbf{r}, \omega + d) - E(\mathbf{r}, \omega - d),$$

d being infinitesimal and perpendicular to Γ at ω.

Our next task is to find the singularities of $D(\omega)$. (We will suppress the dependence on $-i\nabla$ from now on; in fact D may be an integral as well as a differential operator in $\mathbf{r}$.) If there is a pole at ω_n, then as $\omega \to \omega_n$

$$E(\mathbf{r}, \omega) \to i\rho_n(\mathbf{r})/(\omega - \omega_n).$$

But $D(\omega)$ and $S(\mathbf{r}, \omega)$ are analytic, and go to limits $D(\omega_n)$ and $S(\mathbf{r}, \omega_n)$. Hence from (150),

$$D(\omega_n)\rho_n(\mathbf{r}) = 0.$$

We see again that the poles contribute to $E(\mathbf{r}, t)$ a sum of discrete eigensolutions of (150) with $S = 0$.

However, there may be branch lines in E as well as poles. At such a branch line, since $D(\omega)$ and $S(\mathbf{r}, \omega)$ are analytic,

$$D(\omega)E(\mathbf{r}, \omega \pm d) = S(\mathbf{r}, \omega)$$

and taking the difference we find that

$$D(\omega)\rho(\mathbf{r}, \omega) = 0$$

for all ω on Γ. Hence the branch line Γ is the locus of all points ω at which there is a continuum solution of (150) with $S = 0$, and the discontinuity across Γ at ω is the solution.

Our discussion has shown that a branch line in $E(\mathbf{r}, \omega)$, or in other words a continuum of eigensolutions, can occur only for a nonuniform system. On the other hand, from the viewpoint of the homogeneous differential equation for these eigensolutions, a continuum occurs when there exists a singularity for which we do not have sufficient continuity conditions to determine a definite ω. Whenever such a singularity occurs, the integral (156) will distribute it over the region within which the system is nonuniform.